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# Some explicit identities on Changhee-Genocchi polynomials and numbers

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**Abstract**

In this paper, we introduce a new family of functions, which is called the Changhee-Genocchi polynomials. We study some explicit identities on these polynomials, which are related to Genocchi polynomials and Changhee polynomials. Also, we represent Changhee-Genocchi polynomials by gamma and beta functions.

We also study some properties of higher-order Changhee-Genocchi polynomials related to Changhee polynomials and Daehee polynomials.

**MSC:** 05A10; 05A19; 11B68; 11S80

**Keywords:** Euler polynomials; Changhee polynomials; Genocchi polynomials; Changhee-Genocchi numbers; beta and gamma functions

## 1 Introduction

The Genocchi polynomials are defined by the generating function (see [1, 2])

$$\frac{2t}{e^t+1}e^{xt}=\sum_{n}^{\infty}G_n(x)\frac{t^n}{n!}. \tag{1}$$

When $x=0$, $G_n=G_n(0)$ are called the Genocchi numbers. From (1) we see that

$$\begin{aligned}\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!}&=\left(\frac{2t}{e^t+1}\right)e^{xt}=\left(\sum_{l=0}^{\infty}G_l\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}x^m\frac{t^m}{m!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}G_lx^{n-l}\right)\frac{t^n}{n!}.\end{aligned}\tag{2}$$

We consider Changhee-Genocchi polynomials defined by the generating function

$$\frac{2\log(1+t)}{2+t}(1+t)^x=\sum_{n=0}^{\infty}CG_n(x)\frac{t^n}{n!}. \tag{3}$$

When $x=0$, $CG_n=CG_n(0)$ are called the Changhee-Genocchi numbers.

The gamma and beta functions are defined by the following definite integrals:

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1}\,dt, \quad \alpha > 0, \tag{4}$$

and

$$\begin{aligned} B(\alpha,\beta) &= \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt \\ &= \int_0^\infty \frac{t^{\alpha-1}}{(1+t)^{\alpha+\beta}}\,dt, \quad \alpha > 0, \beta > 0. \end{aligned} \tag{5}$$

From (4) and (5) we have (see [3])

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha), \qquad B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{6}$$

We recall that the classical Stirling numbers of the first kind $S_1(n,k)$ and $S_2(n,k)$ are defined by the relations (see [4])

$$\begin{aligned} (x)_n &= \sum_{k=0}^n S_1(n,k)x^k \quad \text{and} \\ x^n &= \sum_{k=0}^n S_2(n,k)(x)_k, \end{aligned}$$

respectively. Here $(x)_n = x(x-1)\cdots(x-n+1)$ denotes the falling factorial polynomial of order $n$. We also have

$$\begin{aligned} \sum_{n=m}^\infty S_2(n,m)\frac{t^n}{n!} &= \frac{(e^t-1)^m}{m!} \quad \text{and} \\ \sum_{n=m}^\infty S_1(n,m)\frac{t^n}{n!} &= \frac{(\log(1+t))^m}{m!}. \end{aligned} \tag{7}$$

In this paper, we introduce a new family of functions, which is called the Changhee-Genocchi polynomials.

We study some properties of these polynomials, which are related to Genocchi polynomials and Changhee polynomials. Also we represent Changhee-Genocchi polynomials by gamma and beta functions.

We also study higher-order Changhee-Genocchi polynomials related to Changhee polynomials and Daehee polynomials.

Most of the ideas in this paper come from Kim and Kim [5]. Specifically, equations (14), (21), and (22) are related to the papers [5–8].

## 2 Changhee-Genocchi polynomials

First, we relate our newly defined Changhee-Genocchi polynomials to Genocchi polynomials.

Replacing $t$ by $e^t - 1$ in (3) and applying (7), we get

$$\begin{aligned}\frac{2t}{e^t+1}e^{tx} &= \sum_{n=0}^{\infty} CG_n(x)\frac{1}{n!}\left(e^t-1\right)^n \\ &= \sum_{n=0}^{\infty} CG_n(x)\frac{1}{n!}n!\sum_{k=n}^{\infty} S_2(k,n)\frac{t^k}{k!} \\ &= \sum_{k=0}^{\infty}\left(\sum_{n=0}^{k} CG_n(x)S_2(k,n)\right)\frac{t^k}{k!}. \end{aligned} \tag{8}$$

The left-hand side of (8) is the generating function of the Genocchi polynomials. Thus, by comparing the coefficients of (1) and (8) we have the following theorem.

**Theorem 1** *For any nonnegative integer $k$, we have*

$$G_k(x) = \sum_{n=0}^{k} CG_n(x)S_2(k,n). \tag{9}$$

On the other hand, if we replace $t$ by $\log(1+t)$ in (1) and apply (7), then we get

$$\begin{aligned}\frac{2\log(1+t)}{2+t}(1+t)^x &= \sum_{n=0}^{\infty} G_n(x)\frac{1}{n!}\left(\log(1+t)\right)^n \\ &= \sum_{n=0}^{\infty} G_n(x)\frac{1}{n!}n!\sum_{k=n}^{\infty} S_1(k,n)\frac{t^k}{k!} \\ &= \sum_{k=0}^{\infty}\left(\sum_{n=0}^{k} G_n(x)S_1(k,n)\right)\frac{t^k}{k!}, \end{aligned} \tag{10}$$

where $S_1(k,n)$ are the Stirling numbers of the first kind.

By comparing the coefficients of both sides of (10), we get the following theorem.

**Theorem 2** *For any nonnegative integer $k$, we have*

$$CG_k(x) = \sum_{n=0}^{k} G_n(x)S_1(k,n). \tag{11}$$

**Remark** When $x = 0$ in (11), we can see that Changhee-Genocchi numbers are integers.

We can consider equation (11) as the inversion formula for (9). From (3) we can consider the following identity:

$$\begin{aligned}\sum_{n=0}^{\infty} CG_n(x)\frac{t^n}{n!} &= \frac{2\log(1+t)}{2+t}(1+t)^x = \left(\sum_{l=0}^{\infty} CG_l\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}(x)_m\frac{t^m}{m!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l} CG_l(x)_{n-l}\right)\frac{t^n}{n!}. \end{aligned} \tag{12}$$

Thus, by comparing the coefficients of both sides of (12) we have

$$\begin{aligned} CG_n(x) &= \sum_{l=0}^{n}\binom{n}{l}CG_l(x)_{n-l} = \sum_{l=0}^{n}\binom{n}{l}CG_{n-l}(x)_l \\ &= \sum_{l=0}^{n}\left(\sum_{m=0}^{n-l}\binom{n}{l}CG_l S_1(n-l,m)x^m\right). \end{aligned} \tag{13}$$

From (13) we can derive the following theorem.

**Theorem 3** *For any nonnegative integer $n$, we have*

$$\int_0^1 CG_n(x)\,dx = \sum_{l=0}^{n}\sum_{m=0}^{n-l}\binom{n}{l}CG_l S_1(n-l,m)\frac{1}{m+1}. \tag{14}$$

In this paper, we define the λ-Changhee-Genocchi polynomials by a generating function as follows:

$$\frac{2\log(1+t)}{(1+t)^{\lambda}+1}(1+t)^{\lambda x} = \sum_{n=0}^{\infty} CG_{n,\lambda}(x)\frac{t^n}{n!}. \tag{15}$$

We recall that the λ-Changhee polynomials are defined in [9] by

$$\frac{2}{(1+t)^{\lambda}+1}(1+t)^{\lambda x} = \sum_{n=0}^{\infty} Ch_{n,\lambda}(x)\frac{t^n}{n!}. \tag{16}$$

When $\lambda = 1$, Changhee-Genocchi polynomials are well-known Changhee polynomials, cf. [10–18]. In order to establish a reflexive symmetry on the Changhee-Genocchi polynomials, we consider the following:

$$\begin{aligned} \sum_{n=0}^{\infty} CG_n(1-x)\frac{t^n}{n!} &= \frac{2\log(1+t)}{1+(1+t)}(1+t)^{1-x} = -\frac{2\log(1+t)}{(1+t)^{-1}+1}(1+t)^{-x} \\ &= \sum_{n=0}^{\infty} CG_{n,-1}(x)\frac{t^n}{n!}. \end{aligned} \tag{17}$$

By comparing the coefficients of (17) we have the following theorem.

**Theorem 4** *For $n \in \mathbb{N}$, we have*

$$CG_n(1-x) = CG_{n,-1}(x). \tag{18}$$

Thus, from (3) and (18) we have

$$\begin{aligned} \sum_{n=0}^{\infty} CG_n\bigl(-x+(1-y)\bigr)\frac{t^n}{n!} &= \frac{2\log(1+t)}{2+t}(1+t)^{-x+(1-y)} \\ &= \frac{2\log(1+t)}{2+t}(1+t)^{-x}(1+t)^{1-y} \end{aligned}$$

$$= \left(\sum_{m=0}^{\infty} CG_m(-x)\frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty}(1-y)_l(-x)\frac{t^l}{l!}\right)$$
$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}CG_m(-x)(1-y)_{n-m}\right)\frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{k=0}^{n-m}\binom{n}{m}CG_m(-x)S_1(n-m,k)(1-y)^k. \tag{19}$$

By comparing the coefficients of (19) we have

$$CG_n\big(1-(x+y)\big) = \sum_{m=0}^{n}\sum_{k=0}^{n-m}\binom{n}{m}CG_m(-x)S_1(n-m,k)(1-y)^k. \tag{20}$$

On the other hand, by (5), (6), and (20) we have

$$\int_0^1 y^n CG_n\big(1-(x+y)\big)\,dy$$
$$= \sum_{m=0}^{n}\sum_{k=0}^{n-m}\binom{n}{m}CG_m(-x)S_1(n-m,k)B(n+1,k+1)$$
$$= \sum_{m=0}^{n}\sum_{k=0}^{n-m}\binom{n}{m}CG_m(-x)S_1(n-m,k)\frac{\Gamma(n+1)\Gamma(k+1)}{\Gamma(n+k+2)}. \tag{21}$$

Thus, by (18) and (21) we have the following identities, which relate the λ-Changhee-Genocchi polynomials, the Stirling numbers, and the beta and gamma polynomials:

$$\int_0^1 y^n CG_{n,-1}(x+y)\,dy$$
$$= -\sum_{l=0}^{n}\sum_{m=0}^{n-l}\binom{n}{l}S_1(n-l,m)CG_l\int_0^1 y^n\big(1-(x+y)\big)^m\,dy$$
$$= -\sum_{l=0}^{n}\sum_{m=0}^{n-l}\sum_{k=0}^{m}\binom{n}{l}\binom{m}{k}S_1(n-l,m)(-x)^{m-k}CG_l\int_0^1 y^n(1-y)^k\,dy$$
$$= -\sum_{l=0}^{n}\sum_{m=0}^{n-l}\sum_{k=0}^{m}\binom{n}{l}\binom{m}{k}S_1(n-l,m)(-x)^{m-k}CG_lB(n+1,k+1)$$
$$= -\sum_{l=0}^{n}\sum_{m=0}^{n-l}\sum_{k=0}^{m}\binom{n}{l}\binom{m}{k}S_1(n-l,m)(-x)^{m-k}CG_l\frac{\Gamma(n+1)\Gamma(k+1)}{\Gamma(n+k+2)}. \tag{22}$$

From (16) we consider

$$\sum_{n=0}^{\infty} CG_{n,\lambda}(1-x)\frac{t^n}{n!} = \frac{2\log(1+t)}{(1+t)^{\lambda}+1}(1+t)^{\lambda(1-x)} = \frac{2\log(1+t)}{1+(1+t)^{-\lambda}}(1+t)^{-\lambda x}$$
$$= \sum_{n=0}^{\infty} CG_{n,-\lambda}(x)\frac{t^n}{n!}. \tag{23}$$

By comparing the coefficients of (23) we have the following theorem.

**Theorem 5** *For any nonnegative integer n, we have*

$$CG_{n,\lambda}(1-x) = CG_{n,-\lambda}(x). \tag{24}$$

**Remark** If we take $\lambda = 1$ in Theorem 5, then we have the result in Theorem 4.

From the second line of (23) and from (16) we have

$$\begin{aligned}
&\left(\sum_{l=1}^{\infty} \frac{(-1)^{l-1}t^l}{l}\right)\left(\sum_{m=0}^{\infty} Ch_{m,\lambda}(x)\frac{t^m}{m!}\right) \\
&\quad = \sum_{n=1}^{\infty}\left(\sum_{l=1}^{n} \frac{(-1)^{l-1}}{l}\frac{Ch_{n-l,\lambda}(x)}{(n-l)!}n!\right)\frac{t^n}{n!}.
\end{aligned} \tag{25}$$

By comparing the coefficients of (23) and (25) we have the following theorem.

**Theorem 6** *For any positive integer n, we have*

$$CG_{n,\lambda}(x) = \sum_{l=1}^{n} \frac{(-1)^{l-1}}{l} Ch_{n-l,\lambda}(x)\frac{n!}{(n-l)!}.$$

For $r \in \mathbb{N}$, we define the Changhee-Genocchi polynomials $CG_n^{(r)}(x)$ of order $r$ by the generating function

$$\left(\frac{2\log(1+t)}{2+t}\right)^r (1+t)^x = \sum_{n=0}^{\infty} CG_n^{(r)}(x)\frac{t^n}{n!}. \tag{26}$$

From (26) we have the following relation between the Changhee-Genocchi polynomials of order $r$ and the Changhee polynomials of order $r$:

$$\begin{aligned}
&\bigl(\log(1+t)\bigr)^r\left(\frac{2}{2+t}\right)^r(1+t)^x \\
&\quad = \left(r!\sum_{l=r}^{\infty} S_2(l,r)\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty} Ch_m^{(r)}(x)\frac{t^m}{m!}\right) \\
&\quad = \left(\sum_{l=0}^{\infty} S_2(l+r,r)\frac{r!t^{l+r}}{(l+r)!}\right)\left(\sum_{m=0}^{\infty} Ch_m^{(r)}(x)\frac{t^m}{m!}\right) \\
&\quad = \left(\sum_{l=0}^{\infty} S_2(l+r,r)\binom{l+r}{r}^{-1}\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty} Ch_m^{(r)}(x)\frac{t^m}{m!}\right)t^r \\
&\quad = \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}S_2(l+r,r)\binom{l+r}{r}^{-1}Ch_{n-l}^{(r)}(x)\right)\frac{t^{n+r}}{n!}.
\end{aligned} \tag{27}$$

By comparing the coefficients of (26) and (27) we have the following theorem.

**Theorem 7** *For any nonnegative integer n, we have*

$$CG_n^{(r)}(x) = \sum_{l=0}^{n}\binom{n}{l}\binom{l+r}{r}^{-1}S_2(l+r,r)Ch_{n-l}^{(r)}(x).$$

For $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$, we have the following identity:

$$\sum_{a=0}^{d-1}(-1)^a(1+t)^a = \frac{1+(1+t)^d}{2+t}. \tag{28}$$

So, for such $d \equiv 1 \pmod 2$, from (28), (3), and (15) we see that

$$\begin{aligned}\sum_{n=0}^{\infty} CG_n(x)\frac{t^n}{n!} &= \frac{2\log(1+t)}{2+t}(1+t)^x \\ &= \sum_{a=0}^{d-1}(-1)^a \frac{2\log(1+t)}{(1+t)^d+1}(1+t)^{d(\frac{a+x}{d})} \\ &= \sum_{a=0}^{d-1}(-1)^a \sum_{n=0}^{\infty} CG_{n,d}\left(\frac{a+x}{d}\right)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{a=0}^{d-1}(-1)^a CG_{n,d}\left(\frac{a+x}{d}\right)\right)\frac{t^n}{n!}. \end{aligned} \tag{29}$$

By comparing the coefficients in (29), for $d \equiv 1 \pmod 2$, we have the following theorem.

**Theorem 8** *For any nonnegative integer $n$ and $d \equiv 1 \pmod 2$, we have*

$$CG_n(x) = \sum_{a=0}^{d-1}(-1)^a CG_{n,d}\left(\frac{a+x}{d}\right). \tag{30}$$

We remark that, for $d \equiv 1 \pmod 2$, from (9) and (30) we have the inversion of Theorem 8.

**Theorem 9** *For any nonnegative integer $n$ and $d \equiv 1 \pmod 2$, we have*

$$\begin{aligned} G_k(x) &= \sum_{n=0}^{k} CG_n(x)S_2(k,n) \\ &= \sum_{n=0}^{k}\left(\sum_{a=0}^{d-1}(-1)^a CG_{n,d}\left(\frac{a+x}{d}\right)S_2(k,n)\right). \end{aligned}$$

From the generating function of the Changhee-Genocchi polynomials in (1), replacing $t$ by $\lambda \log(1+t)$, we get

$$\begin{aligned}\frac{2\lambda\log(1+t)}{(1+t)^\lambda+1}(1+t)^{\lambda x} &= \sum_{n=0}^{\infty} G_n(x)\frac{1}{n!}\bigl(\lambda\log(1+t)\bigr)^n \\ &= \sum_{n=0}^{\infty}\lambda^n G_n(x)\left(\sum_{k=n}^{\infty} S_1(k,n)\frac{t^k}{k!}\right) \\ &= \sum_{k=0}^{\infty}\left(\sum_{n=0}^{k}\lambda^n G_n(x)S_1(k,n)\right)\frac{t^k}{k!}. \end{aligned} \tag{31}$$

Thus, the left-hand side of (31) can be represented by the λ-Changhee-Genocchi polynomials as follows:

$$\frac{2\lambda\log(1+t)}{(1+t)^{\lambda}+1}(1+t)^{\lambda x}=\lambda\sum_{k=0}^{\infty}CG_{k,\lambda}(x)\frac{t^k}{k!}. \tag{32}$$

By comparing the coefficients of (31) and (32) we have the following theorem.

**Theorem 10** *For any nonnegative integer $k$, we have*

$$CG_{k,\lambda}(x)=\sum_{n=0}^{k}\lambda^{n-1}G_n(x)S_1(k,n).$$

From the generating function of the Changhee-Genocchi numbers in (3) we want to see the recurrence relation for the Changhee-Genocchi numbers:

$$\begin{aligned}2\log(1+t)&=\sum_{n=0}^{\infty}CG_n\frac{t^n}{n!}(t+2)\\&=\sum_{n=1}^{\infty}CG_n\frac{t^{n+1}}{n!}+\sum_{n=0}^{\infty}2CG_n\frac{t^n}{n!}\\&=\sum_{n=2}^{\infty}nCG_{n-1}\frac{t^n}{n!}+2\sum_{n=1}^{\infty}CG_n\frac{t^n}{n!}\\&=2CG_1t+\sum_{n=2}^{\infty}(nCG_{n-1}+2CG_n)\frac{t^n}{n!}.\end{aligned} \tag{33}$$

On the other hand, from the left-hand side of (33) we have

$$2\log(1+t)=\sum_{n=1}^{\infty}(-1)^{n-1}2(n-1)!\frac{t^n}{n!}. \tag{34}$$

By comparing the coefficients of (33) and (34) we have the following recurrence relation for the Changhee-Genocchi numbers.

**Theorem 11** *We have*

$$\begin{aligned}&CG_0=0,\\&nCG_{n-1}+2CG_n=2(n-1)!(-1)^{n-1}\quad\textit{for } n\geq 1.\end{aligned}$$

From the higher-order Changhee-Genocchi polynomials

$$\left(\frac{2\log(1+t)}{2+t}\right)^r(1+t)^x=\sum_{n=0}^{\infty}CG_n^{(r)}(x)\frac{t^n}{n!} \tag{35}$$

we can deduce

$$CG_0^{(r)}(x)=CG_1^{(r)}(x)=\cdots=CG_{r-1}^{(r)}(x)=0. \tag{36}$$

Thus, from (36) we can rewrite (35) as follows:

$$\left(\frac{2\log(1+t)}{2+t}\right)^r (1+t)^x = \sum_{n=0}^{\infty} CG_{n+r}^{(r)}(x)\frac{t^{n+r}}{(n+r)!}. \tag{37}$$

We recall that the Dahee polynomials are defined by the generating function (see [9, 19])

$$\frac{\log(1+t)}{t}(1+t)^x = \sum_{n=0}^{\infty} D_n(x)\frac{t^n}{n!}.$$

When $x = 0$, $D_n = D_n(0)$ are called the Dahee numbers.

For $r \in \mathbb{N}$, the higher-order Daehee numbers are given by the generating function (see [9, 19, 20])

$$\left(\frac{\log(1+t)}{t}\right)^r = \sum_{n=0}^{\infty} D_n^{(r)}(x)\frac{t^n}{n!}.$$

From (28) we have

$$\begin{aligned}
2\log(1+t)\sum_{a=0}^{d-1}(-1)^a(1+t)^a &= \frac{2\log(1+t)}{2+t} + \frac{2\log(1+t)}{t+2}(1+t)^d \\
&= \frac{2\log(1+t)}{t}\left(\sum_{a=0}^{d-1}(-1)^a(1+t)^a\right) \\
&= \sum_{n=0}^{\infty} CG_n\frac{t^{n-1}}{n!} + \sum_{n=0}^{\infty} CG_n(d)\frac{t^{n-1}}{n!} \\
&= \sum_{n=0}^{\infty}\left(2\sum_{a=0}^{d-1}(-1)^a D_n(a)\right)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\frac{CG_{n+1}}{n+1} + \frac{CG_{n+1}(d)}{n+1}\right)\frac{t^n}{n!}.
\end{aligned} \tag{38}$$

Thus, from (38) we have the following theorem.

**Theorem 12** *For any nonnegative integer n and* $d \equiv 1 \pmod 2$, *we have*

$$2\sum_{a=0}^{d-1}(-1)^a D_n(a) = \frac{CG_{n+1}}{n+1} + \frac{CG_{n+1,d}}{n+1}.$$

## 3 Changhee-Genocchi polynomials arising from differential equations

In this section, we give new identities on the Changhee-Genocchi numbers by using differential equations. We use the idea recently developed by Kwon et al. [21].

By equation (3) we can write the generating function for the Changhee-Genocchi numbers as follows:

$$F(t) = \frac{2\log(1+t)}{2+t} = \sum_{n=0}^{\infty} CG_n\frac{t^n}{n!}. \tag{39}$$

Let

$$G(t) = \log(1+t) \quad \text{and} \quad H(t) = \frac{2}{2+t}.$$

Then

$$\begin{aligned} G^{(N)}(t) &= \left(\frac{d}{dt}\right)^N G(t) = (-1)^{N-1}(N-1)!e^{-N\cdot G(t)}, \quad \text{and} \\ H^{(N)}(t) &= \left(\frac{d}{dt}\right)^N H(t) \\ &= \left(-\frac{1}{2}\right)^N N!e^{-(N+1)\cdot K(t)}, \quad \text{where } K(t) = \log(1+t/2). \end{aligned}$$

Thus,

$$\begin{aligned} F^{(N)}(t) &= \left(\frac{d}{dt}\right)^N F(t) = \sum_{k=0}^{N}\binom{N}{k}G^{(N-k)}H^{(k)} \\ &= \sum_{k=0}^{N}\binom{N}{k}(-1)^{N-k-1}(N-k-1)!e^{-(N-k)G(t)} \\ &\quad \times \left(-\frac{1}{2}\right)^k k!e^{-(k+1)K(t)} \\ &= \sum_{k=0}^{N}\binom{N}{k}(-1)^{N-1}\left(\frac{1}{2}\right)^k k!(N-k-1)!e^{-(N-k)G(t)}e^{-(k+1)K(t)}. \end{aligned} \tag{40}$$

On the other hand,

$$\begin{aligned} e^{-(N-k)G}e^{-(k+1)K} &= \left(\sum_{n=0}^{\infty}(-N+k)^n\frac{G^n}{n!}\right)\left(\sum_{l=0}^{\infty}\bigl(-(k+1)\bigr)^l\frac{K^l}{l!}\right) \\ &= \left(\sum_{n=0}^{\infty}(-N+k)^n\sum_{m=n}^{\infty}S_1(m,n)\frac{t^m}{m!}\right) \\ &\quad \times \left(\sum_{l=0}^{\infty}(-k-1)^l\sum_{j=l}^{\infty}\frac{1}{2^j}S_1(j,l)\frac{t^j}{j!}\right) \\ &= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}(-N+k)^nS_1(m,n)\right)\frac{t^m}{m!} \\ &\quad \times \sum_{j=0}^{\infty}\left(\sum_{l=0}^{j}(-k-1)^lS_1(j,l)\frac{1}{2^j}\right)\frac{t^j}{j!} \\ &= \sum_{s=0}^{\infty}\left(\sum_{m=0}^{s}\binom{s}{m}\sum_{n=0}^{m}(-N+k)^nS_1(m,n)\right. \\ &\quad \left.\times \sum_{l=0}^{s-m}(-k-1)^lS_1(s-m,l)\frac{1}{2^{s-m}}\right)\frac{t^s}{s!}. \end{aligned} \tag{41}$$

From (39) we have

$$F^{(N)}(t) = \left(\frac{d}{dt}\right)^N F(t) = \sum_{m=0}^{\infty} CG_{N+m}\frac{t^m}{m!}. \tag{42}$$

By comparing the coefficients of (40), (41), and (42) we have new identities on the Changhee-Genocchi numbers as follows.

**Theorem 13** *For any nonnegative integer s, we have*

$$\begin{aligned} CG_{s+N} = {} & \sum_{m=0}^{s}\binom{s}{m}\left\{\left(\sum_{n=0}^{m}(-N+k)^n S_1(m,n)\right)\left(\sum_{l=0}^{s-m}(-k-1)^l S_1(s-m,l)\frac{1}{2^{s-m}}\right)\right\} \\ & \times \sum_{k=0}^{N}\binom{N}{k}(-1)^{N-1}\left(\frac{1}{2}\right)^k k!(N-k-1)!. \end{aligned}$$

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed equally to this work. All authors read and approved the final manuscript.

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